Combinatorial Geometry

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1 Extremal Principle and Induction

In many combinatorial geometry problems, our task is to find an object with a certain property P. Oftentimes, taking the object that instead maximizes a quantity Q can magically solve the problem – the maximizer of Q can be shown simply to have the property P. The difficult aspect of applying this technique, known as the *extremal principle*, is devising the quantity Q. We'll start with a classical but very tricky example.

Example 1 (Sylvester-Gallai). *Given a finite set S of points in a plane, either they all lie on a line or there is a line containing exactly two points of S.*

Solution. Assume for contradiction that the points of S do not all lie on a line and there is no line containing exactly two points of S. Consider the triple (A, B, C) of distinct points in S that are not collinear and such that the distance from A to the line ℓ passing through B and C is minimal. Since S is finite and not all contained in a line, this triple exists. Since there is no line passing through exactly two points of S, there is a point $D \neq B, C$ on ℓ. Let P be the projection of A onto ℓ and note that two of B, C and D lie on the same side of P on ℓ. Without loss of generality, assume these are C and D and that C is closer to P than D. Let P' be the projection of C onto AD. Note that $\triangle DCP'$ is similar to $\triangle DAP$ with ratio less than 1. This implies that $AP > CP'$ and therefore the triple (C, A, D) contradicts the minimality of (A, B, C). This shows that there could not have been a third point D on ℓ, completing the proof. □

At first glance, this solution feels wildly unmotivated. Why choose the smallest distance to a line? Why use the extremal principle at all? The extremal principle is just a clean way of writing up a proof that uses *greedy algorithm* to find P by trying to optimize a well-chosen quantity Q. You can think of the proof above as saying that if a line ℓ I am looking at has at least three points on it (i.e. does not have property P = contains two points) then I can find another line with a point closer to it than any points were to ℓ, which in this case was the line AD. Since the point set is finite, this greedy algorithm clearly can't run forever and eventually will find a line containing two points for us. At the heart of these proofs is the following question:

Question: If the object I am looking at does not have property P, can I come up with a quantity Q that I can always make larger (or smaller)?

To answer this question, we should be trying to exploit some local condition like the existence of a third point in the example above. In the next example, this quantity Q is the sum of the lengths of the segments in the desired pairing. It in fact directly shows that the natural greedy algorithm of simply rewiring intersections terminates.

Example 2. *Given $2n$ points in a plane with no three collinear, with n red points and n blue points, prove that there exists a pairing of the red and blue points such that then segments joining each pair are pairwise non-intersecting.*

Solution. Consider the pairing with smallest sum of the lengths of the segments. Assume for contradiction that some pair of segments AC and BD in the pairing that intersects. Let P be the intersection of AC and BD where A and D are red and B and C are blue. By the triangle inequality,

$$AB < AP + BP \quad \text{and} \quad CD < CP + DP$$

Therefore $AB + CD < AC + BD$. This implies that the matching replacing AC and BD with AB and CD has a lower sum of segments, which is a contradiction. Thus this matching has no pairs of segments that intersect. □

Less sophisticated applications of the extremal principle can also be very useful in combinatorial geometry problems. Some good objects to consider include:

- the longest and shortest segments AB joining points in the set S
- the largest triangle ABC in S (e.g. in terms of area, circumradius, height, etc.)
- the points of minimum/maximum x or y coordinates

The goal is to push some local condition in the problem to its limits. In the next example, taking the longest segment when combined with the condition that the area of the triangle formed by any three points is at most A easily yields useful structure about the overall point set.

Example 3 (CMO 2017)**.** *There are n circles of radius one positioned in a plane such that the area of any triangle formed by the centers of three of these circles is at most A. Prove that there is a line intersecting at least $\frac{n}{1+\sqrt{A}}$ of these circles.*

Solution. Let S be the set of centers of the n circles. We will first show that there is a line ℓ such that the projections of the points in S lie in an interval of length at most $2\sqrt{A}$ on ℓ. Let A and B be the pair of points in S that are farthest apart and let the distance between A and B be d. Now consider any point $C \in S$ distinct from A and B. The distance from C to the line AB must be at most $\frac{2A}{d}$ since triangle ABC has area at most A. Therefore if ℓ is a line perpendicular to AB, then the projections of S onto ℓ lie in an interval of length $\frac{4A}{d}$ centered at the intersection of ℓ and AB. Furthermore, all of these projections must lie on an interval of length at most d on ℓ since the largest distance between two of these projections is at most d. Since $\min(d, 4A/d) \le 2\sqrt{A}$, this proves the claim.

Now note that the projections of the n circles onto the line ℓ are intervals of length 2, all contained in an interval of length at most $2\sqrt{A} + 2$. Each point of this interval belongs to on average $\frac{2n}{2\sqrt{A}+2} = \frac{n}{1+\sqrt{A}}$ of the subintervals of length 2 corresponding to the projections of the n circles onto ℓ. Thus there is some point $x \in \ell$ belonging to the projections of at least $\frac{n}{1+\sqrt{A}}$ circles. The line perpendicular to ℓ through x has the desired property. □

The extremal principle can also be very useful when combined with induction. Often choosing an extremal object to remove can nicely fit into an induction argument as in the next example.

Example 4. *There are finitely many congruent, parallel squares in a plane such that among any* $k+1$ *squares some two intersect. Show that the squares can be divided into at most* $2k-1$ *nonempty groups such that all squares in the same group have a common point.*

Solution. We prove the claim by induction on k. Consider the square A with minimum y-coordinate and the set S of all squares that intersect A. Note that all squares in S must contain one of the two upper vertices of A. Now consider removing A and all squares in S. Assume for contradiction that there are k remaining squares such that no two intersect. Appending A to these k squares gives $k+1$ that are pairwise disjoint, contradicting the condition in the problem. Thus among any k remaining squares, some two intersect, and by the induction hypothesis there are $2k-3$ points such that all squares not removed pass through at least one of these points. Adding the two upper vertices of A to these points completes the induction.

We now show the base case $k = 1$. Consider the square B with minimal y-coordinate. Every other square passes through an upper vertex of B. If every square passes through one of these vertices, we are done. Otherwise, take the square C of minimum x-coordinate through the upper left vertex of B and the square D of maximum x-coordinate through the upper right vertex of B. These have a common point on B which all squares pass through. □

2 Convex Hulls, Orderings, Processes and Discrete Continuity

A key object in many combinatorial geometry problems is the convex hull. A set $S \subseteq \mathbb{R}^2$ is convex if for any $A, B \in S$, the segment AB is contained in S. The convex hull of a set $S \subseteq \mathbb{R}^2$ is the smallest convex set containing S. If S is a finite point set, then there is a subset $C \subseteq S$ such that the convex hull of S is a convex polygon with vertices C. Considering the convex hull of a set of points can often be a good starting point in geometric problems involving many points.

Another recurring idea is to introduce an order on the points in S induced by some natural process. Examples of common processes and orders include:

- the order in which points are hit by moving a line ℓ with a fixed orientation continuously over S, which James calls a *linesweeper*
- the order in which points by rotating a line ℓ through a fixed point A
- the order in which points are hit by a circle C passing through two fixed points A and B, being deformed by moving its center O continuously along the perpendicular bisector of AB, which I call a *circlesweeper*

There are many other useful processes, often tailored to the problem in question. A quantity Q is discretely continuous if it changes by at most 1 along an ordering of the points. When this is the case, as long as Q begins less than c and ends at least c, it must at some point be exactly c. This can be useful in existence proofs.

The next example combines all of these ideas. The idea is to continuously rotate a line passing through a fixed point on the convex hull of the point set and show that at some point it becomes a balancing line using discrete continuity.

Example 5 (USAMO 2005)**.** *Let* n *be an integer greater than 1. Suppose* $2n$ *points are given in the plane, no three of which are collinear. Suppose* n *of the given* $2n$ *points are colored blue and* *the other n colored red. A line in the plane is called a balancing line if it passes through one blue and one red point and, for each side of the line, the number of blue points on that side is equal to the number of red points on the same side. Prove that there exist at least two balancing lines.*

Solution. Let S be the set of n points and $C \subseteq S$ be the set of vertices of the convex hull of S. Suppose that some two consecutive points along C have different colours. Then the line through these two points has no points on one side and $n-1$ of each colour on the other. If C contains both colours, then there are at least two pairs of consecutive points of different colours and the claim is proven. Thus we can assume that all points on C are one colour, say blue.

Let A be a point on C and let $R_1, R_2, \dots, R_n$ be the red points of S in clockwise order about the center A. Given a line ℓ, let $f(\ell)$ be the number of red points to the right of ℓ minus the number of blue points to the right of ℓ. Since the are no red points to the right of AR_1 and at least one blue point from C to the right of AR_1, it holds that $f(AR_1) < 0$. Also note that all of the remaining $n-1$ red points are on the right of RA_n and thus $f(AR_n) \geq 0$. Let j be the first index with $f(AR_j) \geq 0$. Since the number of red points on the right increases by at most 1 from AR_{j-1} to AR_j, it holds that $f(AR_j) - f(AR_{j-1}) \leq 1$. Thus it must hold that $f(AR_j) = 0$ and AR_j is a balancing line. Therefore there is at least one balancing line through each point in C. Since $|C| \geq 3$, we are done. □

The next two examples have a common setup. There is a set S of $2n+1$ points in a plane such that no four are concyclic and no three are collinear. A circle is called *halving* if it passes through three of these points and contains exactly $n-1$ points in its interior. We use several different techniques to prove sequentially stronger statements about the number of halving circles of S.

Example 6. *There are at least $\frac{2n^2+n}{3}$ halving circles of S and the number of halving circles of S has the same parity as n.*

Solution. Fix a pair of points A and B and let $N(A,B)$ be the number of halving circles through A and B. Consider a circlesweeper passing through A and B. In other words, let O begin far down on the perpendicular bisector of A and B and continuously move it until it is far up on this bisector. Consider the circle centered at O and passing through A and B. Let $f(O)$ denote the number of points inside this circle. If x is the number of points below the line AB, then $f(O)$ begins at x and ends at $2n-1-x$. Each time $f(O)$ changes, it changes by exactly 1 since it either gains or loses a point in its interior and no four points are concyclic. Note that $N(A,B)$ is the number of times $f(O)$ changes to $n-1$. Since $f(O)$ changes sides of $n-1$, by discrete continuity there is some first O when it changes to $n-1$. At this O, it must pass through three points of S. Therefore $N(A,B) \geq 1$. We thus have that the number of halving circles is as least

$$\frac{1}{3} \sum_{A<B} N(A,B) \geq \frac{1}{3} \binom{2n+1}{2} = \frac{2n^2+n}{3}$$

since each halving circle is counted three times in the sum above. □

Carefully examining how $f(O)$ increases and decreases and its parity can be used to show that $N(A,B)$ is odd for all distinct A and B. This can then be used to prove the following intriguing property of the number of halving circles from APMO 1999, which we leave as an exercise.

Example 7 (APMO 1999)**.** *The number of halving circles of S has the same parity as n.*

In fact something much stronger holds for the number of halving circles. As proven by Ardila, there are always exactly n^2 halving circles. With combinatorial properties that are invariant to the underlying point set S, it is often a useful to view their statements *differentially* i.e. by trying to change the point set S and show the property is still true. We sketch a proof of this fact with this approach.

Example 8. *There are exactly n^2 halving circles of S.*

Solution Sketch. Consider moving a single point P in S along a path and suppose that it has just crossed the circumcircle of $A, B, C \in S$. Some case analysis shows that the number of halving circles among the four circles $(ABC), (ABP), (ACP)$ and (BCP) has remained constant. Thus the number of halving circles remains constant on moving the points of S. This common count can be shown to be n^2 by induction on n. Take slightly perturbed vertices of a regular $(2n-1)$-gon, its center O and a distant point Q. Applying the induction hypothesis yields $(n-1)^2$ halving circles among the vertices of the $(2n-1)$-gon. It can be shown that there are exactly $2n-1$ halving circles passing through at least one of O and Q, completing the proof. □

3 Pigeonhole, Averaging and the Probabilistic Method

Many combinatorial geometry problems involve the pigeonhole principle. The next example is a nice problem demonstrating the power of asking whether a stronger discrete version of a combinatorial geometry problem is also true.

Example 9. *The points of the plane are coloured by finitely many colours. Prove that one can find a rectangle with vertices of the same colour.*

Solution. Suppose that there are n distinct colours in the plane. Consider an $(n+1) \times (n^{n+1}+1)$ grid of points in the plane. There are at exactly n^{n+1} ways to colour the $n+1$ points in a column of the grid. Therefore, by pigeonhole, some two columns are identically coloured. Furthermore, this column pattern must have at least one pair of points of the same colour. This yields a rectangle of points with the same colour. □

When trying to show the existence of a geometric object with a property P, it can often be useful to consider how an *average* or *random* object fares. Quantities can often be easier to handle on average, as the next example shows. One warning is that when choosing an object randomly in a plane or space, there are often analysis-related concerns that one needs to deal with to be completely rigorous. For example, what does a random line mean? In this note, we avoid these issues and instead focus on intuition. In the next example, constructing such a line would be exceedingly painful, if tractable at all. However, on average a random line has the desired property and thus such a line must exist.

Example 10. *Let A and B be two finite sets of segments in three-dimensional space such that the sum of the lengths of the segments in A is larger than the sum of the lengths of the segments in B. Prove that there is a line in space with the property that the sum of the lengths of the projections of the segments in A onto that line is greater than the sum of the lengths of the projections of the segments in B.*

Solution Sketch. Consider the average $E(s)$ of the lengths of the projections of a segment s in three-dimensional space over all possible lines. Since the projection of s onto a fixed line scales linearly with the length $|s|$, so does this average. Thus $E(s) = c|s|$ for some constant $c > 0$. Now note that

$$\sum_{s\in A} E(s) - \sum_{s\in B} E(s) = c\left(\sum_{s\in A} |s| - \sum_{s\in B} |s|\right) > 0$$

and thus the sum of the lengths of the projections of A minus the sum of the lengths of the projections of B is positive on average. Hence it must be positive for a specific line, proving the result. □

Our last two examples illustrate how choosing an object randomly can be immensely powerful. Notably, we will choose a random object not uniformly at random, but instead with carefully chosen weights.

Example 11 (Crossing Numbers). *Given n points in a plane, m paths are drawn connecting pairs of these points. If $m \geq 4n$, then there are at least $m^3/64n^2$ pairs of intersecting paths.*

Given a graph G with vertices in the plane $\mathbb{R}^2$, let $\text{cr}(G)$ be the minimum number of pairs of intersection paths when the edges of G are drawn as paths in the plane. We then derive the following weak lower bound using the Euler characteristic.

Lemma 1. *For any graph G with vertices in $\mathbb{R}^2$, it holds that*

$$cr(G) \geq |E(G)| - 3|V(G)| + 6$$

Proof. Consider a drawing of G with $\text{cr}(G)$ crossings. Let m be the minimum edges that need to erased to yield a graph G' with no crossings. Note that $m \leq \text{cr}(G)$ since deleting one edge per crossing removes all of the crossings. After these deletions, the resulting graph is planar. Since each edge is contained in at most two faces and each face uses at least three edges, it follows that $3|F(G')| \leq 2|E(G')|$. Thus Euler's characteristic implies that

$$2 \leq |V(G)| + |F(G')| - |E(G')| \leq |V(G)| - \frac{1}{3}|E(G')| = |V(G)| - \frac{1}{3}\left(|E(G)| - m\right)$$

Rearranging and using $m \leq \text{cr}(G)$ proves the lemma. □

Using a clever application of the probabilistic method, we can boost this weak lower bound to the desired stronger lower bound. The idea is that a mildly bad graph G can be shown to contain a very bad small subgraph H. The key is to pick H randomly by keeping each vertex of G independently with probability p.

Solution to Example 10. Let $N(H)$ be the number of crossings in the drawing of the subgraph H of the n points in G when G is drawn to have cr(G) crossings. By the lemma, we have that

$$N(H) - |E(H)| + 3|V(H)| \geq 6$$

for all subgraphs H. Consider a random subgraph chosen by including each of the n vertices independently with probability p. The average value of $N(H)$ is $\mathbb{E}[N(H)] = p^4 \cdot \text{cr}(G)$ since each crossing of G must have all four of its endpoints included to be in H. The average value of $|V(H)|$

and $|E(H)|$ are similarly $\mathbb{E}[|V(H)|] = pn$ and $\mathbb{E}[|E(G)|] = p^2m$. Therefore the average value of $N(H) - |E(H)| + 3|V(H)|$ is

$$\mathbb{E}[N(H) - |E(H)| + 3|V(H)|] = p^4 \cdot \mathrm{cr}(G) - p^2m + pn \geq 6 > 0$$

This inequality holds since this quantity is at least 6 for any subgraph H and thus must be on average. Setting $p = 4n/m$ and rearranging yields that

$$\mathrm{cr}(G) > \frac{m}{p^2} - \frac{3n}{p^3} = \frac{m^3}{64n^2}$$

which completes the proof. □

This example yields a nice simple proof of a famous theorem from combinatorial geometry.

Example 12 (Szemeredi-Trotter)**.** *Given m distinct lines and n distinct points in a plane, the number of pairs of points and lines such that the point lies on the line is at most $4(m^{2/3}n^{2/3}+m+n)$.*

Solution. Let P be the set of points, L be the set of lines and let I be the number of such pairs. Consider the graph G with vertex set P and edges drawn to be the segments joining consecutive vertices in P along the lines in L. If a line $\ell \in L$ contains $f(\ell)$ vertices of P, then it contains $f(\ell) - 1$ edges of G. Thus

$$|E(G)| = \sum_{\ell \in L} (f(\ell) - 1) = \sum_{\ell \in L} f(\ell) - m = I - m$$

The number of crossings in this drawing of G is at most the number of intersections of pairs of lines in L which is at most $\binom{m}{2}$. By the previous example, either $I - m = |E(G)| < 4n$ or

$$\binom{m}{2} \geq \mathrm{cr}(G) \geq \frac{(I-m)^3}{64n^2}$$

Rearranging yields that in either case, $I \leq 4(m^{2/3}n^{2/3} + m + n)$. □

4 Some Recurring Theorems

The following are several recurring theorems in combinatorial geometry problems.

1. *Erdos-Szekeres*: Any sequence of distinct real numbers with length at least $(r-1)(s-1)+1$ contains a monotonically increasing subsequence of length r or a monotonically decreasing subsequence of length s.

2. *Euler's Characteristic*: If G is a finite connected planar graph, then

$$|V(G)| + |F(G)| - |E(G)| = 2$$

   where $V(G), F(G)$ and $E(G)$ are the numbers of vertices, faces and edges of G, respectively.

3. *Sperner's Lemma*: Given a triangle ABC, and a triangulation T of the triangle, the set S of vertices of T is colored with three colors in such a way that:

- A, B, and C are colored 1, 2, and 3, respectively; and
- each vertex on one of the three edges of ABC is colored with one of the two colors of the ends of its edge.

Then T contains an odd number of triangles with all three distinct colors at their vertices.

4. *Helly's Theorem*: If $X_1, X_2, \dots, X_n$ are convex subsets of $\mathbb{R}^d$ with $n > d$ such that every $d+1$ of these subsets have a nonempty intersection, then all n have a nonempty intersection.

5. *Pick's Theorem*: On the Cartesian plane, given a simple polygon with vertices on lattice points, let x be number of lattice points interior of the polygon and i be the number of lattice points on the boundary of the polygon. Then the area of the polygon is $x + i/2 - 1$.

5 Problems

1. Finitely many lines divide the plane into regions. Show that these regions can be colored by two colors in such a way that neighboring regions have different colors.

2. In a regular $2n$-gon, n diagonals intersect at a point S, which is not a vertex. Prove that S is the center of the $2n$-gon.

3. Let S be a set of n points on a plane in general position. Prove that the points in S can be labeled $P_1, P_2, \dots, P_n$ such that the broken line $P_1P_2 \cdots P_n$ does not intersect itself.

4. Prove that if a convex polygon lies in the interior of another convex polygon, then the perimeter of the inner polygon is less than the perimeter of the outer polygon.

5. There are k points, $2 \le k \le 50$, inside a convex 100-sided polygon. Prove that we can choose at most $2k$ vertices from this 100-sided polygon so that the k points are inside the polygon with the chosen points as vertices.

6. The vertices of a convex polygon are colored by at least three colors such that no two consecutive vertices have the same color. Prove that one can dissect the polygon into triangles by diagonals that do not cross and whose endpoints have different colors.

7. Find all natural numbers n for which a convex n-gon can be divided into triangles by diagonals with disjoint interiors, such that each vertex of the n-gon is the endpoint of an even number of the diagonals.

8. Let $P = P_1P_2 \cdots P_n$ be a convex polygon in the plane such that for any $i \neq j$ there exists some k such that $\angle P_iP_kP_j = 60°$. Prove that P is an equilateral triangle.

9. Given nine points inside the unit square, prove that some three of them form a triangle whose area does not exceed $\frac{1}{8}$.

10. Given is a finite set of spherical planets, all of the same radius and no two intersecting. On the surface of each planet consider the set of points not visible from any other planet. Prove that the total area of these sets is equal to the surface area of one planet.

11. Consider a planar region of area 1, obtained as the union of finitely many disks. Prove that from these disks we can select some that are mutually disjoint and have total area at least $\frac{1}{9}$.

12. Let n and k be positive integers and let S be a set of n points in the plane such that no three points of S are collinear, and for every point P of S there are at least k points of S equidistant from P. Prove that $k < \frac{1}{2} + \sqrt{2n}$.

13. The set M consists of n points in the plane, no three lying on a line. For each triangle with vertices in M, count the number of points of M lying in its interior. Prove that the arithmetic mean of these numbers does not exceed $n/4$.

14. Let $P_1, P_2, \dots, P_{2n}$ be a permutation of the vertices of a regular $2n$-gon. Prove that some two segments P_iP_{i+1} are parallel where $P_{2n+1} = P_1$.

15. A circle is divided into n arcs by n marked points. The circle is then rotated by an angle of $2\pi k/n$ for some positive integer k and the marked points move to n new points, dividing the circle into n new arcs. Prove that there is a new arc that lies entirely in the one of the old arcs.

16. Given a set S of n points in the plane, prove there exists at least $\sqrt{n}$ points in S such that no three are vertices of an equilateral triangle.

17. A set of points is marked on the plane, with the property that any three marked points can be covered with a disk of radius 1. Prove that the set of all marked points can be covered with a disk of radius 1.

18. 500 points are given inside a unit square. Prove that there are some 12 points $A_1, A_2, \dots, A_{12}$ such that $A_1A_2 + A_2A_3 + \cdots + A_{11}A_{12} < 1$.

19. Four congruent right triangles are given. One can cut one of them along the altitude and repeat the operation several times with the newly obtained triangles. Prove that no matter how we perform the cuts, we can always find among the triangles two that are congruent.

20. There are n points inside a right triangle with hypotenuse h. Prove that these points can be labelled $A_1, A_2, \dots, A_n$ so that $A_1A_2^2 + A_2A_3^2 + \cdots + A_{n-1}A_n^2 \le h^2$.

21. For any convex n-gon, there exists $n-2$ points in the interior such that every triangle formed by three vertices of the n-gon contains at exactly one of these points in its interior.

22. Each of n black squares and n white squares can be obtained by a translation from each other. Every two squares of different colours have a common point. Prove that there is a point belonging at least to n squares.

23. Find all finite set of points S, with no three collinear, such that if $A, B, C \in S$ then there exists $D \in S$ such that $A, B, C, D \in S$ are the vertices of a parallelogram.

24. There are two families of convex polygons in the plane. Each family has a pair of disjoint polygons. Any polygon from one family intersects any polygon from the other family. Show that there is a line which intersects all the polygons.

25. Let $\mathcal{S}$ be a finite set of at least two points in the plane. Assume that no three points of $\mathcal{S}$ are collinear. A windmill is a process that starts with a line ℓ going through a single point $P \in \mathcal{S}$. The line rotates clockwise about the pivot P until the first time that the line meets some other point belonging to $\mathcal{S}$. This point, Q, takes over as the new pivot, and the line now rotates clockwise about Q, until it next meets a point of $\mathcal{S}$. This process continues indefinitely. Show that we can choose a point P in $\mathcal{S}$ and a line ℓ going through P such that the resulting windmill uses each point of $\mathcal{S}$ as a pivot infinitely many times.

26. Let S be a finite set of points in three-dimensional space. Let S_x, S_y, S_z be the sets consisting of the orthogonal projections of the points of S onto the yz-plane, zx-plane, xy-plane, respectively. Prove that
$$|S|^2 \leq |S_x| \cdot |S_y| \cdot |S_z|$$

27. In the interior of the convex 2011-gon are 2011 points, such that no three among the given 4022 points (the interior points and the vertices) are collinear. The points are coloured one of two different colours and a colouring is called *good* if some of the points can be joined in such a way that the following conditions are satisfied:

    - each segment joins two points of the same colour;
    - none of the line segments intersect; and
    - for any two points of the same colour there exists a path of segments connecting them.

    Find the number of good colourings.

28. Show that from any finite collection of closed hemispheres that cover a sphere one can choose four that cover the sphere.

29. Consider a square of sidelength n and $(n+1)^2$ interior points. Prove that we can choose 3 of these points so that they determine a triangle of area at most $\frac{1}{2}$.

30. Let n be a natural number. Suppose A and B are two sets, each containing n points in the plane, such that no three points of a set are collinear. Let $T(A)$ be the number of broken lines with vertices from A, containing $n-1$ segments and that do not intersect themselves. Define $T(B)$ similarly. If the points of B are vertices of a convex n-gon but the points of A are not, prove that $T(B) < T(A)$.

31. On the cartesian plane are drawn several rectangles with the sides parallel to the coordinate axes. Assume that any two rectangles can be cut by a vertical or a horizontal line. Show that it's possible to draw one horizontal and one vertical line such that each rectangle is cut by at least one of these two lines.

32. On a plane are given finitely many red and blue lines, no two parallel, such that any intersection point of two lines of the same color also lies on another line of the other color. Prove that all the lines pass through a single point.

33. Let M be a set of $n \geq 4$ points in the plane, no three of which are collinear. Initially these points are connected with n segments so that each point in M is the endpoint of exactly two segments. Then, at each step, one may choose two segments AB and CD sharing a common

interior point and replace them by the segments AC and BD if none of them is present at this moment. Prove that it is impossible to perform $n^3/4$ or more such moves.

34. There are n circles drawn on a piece of paper in such a way that any two circles intersect in two points, and no three circles pass through the same point. Turbo the snail slides along the circles in the following fashion. Initially he moves on one of the circles in clockwise direction. Turbo always keeps sliding along the current circle until he reaches an intersection with another circle. Then he continues his journey on this new circle and also changes the direction of moving, i.e. from clockwise to anticlockwise or *vice versa*. Suppose that Turbo?s path entirely covers all circles. Prove that n must be odd.

35. Given is a convex polygon P with n vertices. Triangle whose vertices lie on vertices of P is called good if all its sides are equal in length. Prove that there are at most $\frac{2n}{3}$ good triangles.

36. There are 2017 mutually external circles drawn on a blackboard, such that no two are tangent and no three share a common tangent. A tangent segment is a line segment that is a common tangent to two circles, starting at one tangent point and ending at the other one. Luciano is drawing tangent segments on the blackboard, one at a time, so that no tangent segment intersects any other circles or previously drawn tangent segments. Luciano keeps drawing tangent segments until no more can be drawn. Find all possible numbers of tangent segments when Luciano stops drawing.

37. Show that it is impossible to cover a unit square with five equal squares with side $s < \frac{1}{2}$.

38. Given n points in a plane, prove that it is possible to draw a sector with angle $2\pi/n$ at each point so that the entire plane is covered.